4.7: Exponential and Logarithmic Models

Exponential Growth

- Exponential Growth is modeled by function $n(t) = n_0 e^{rt}$, where

  $n(t)$ is population at time t

  n_0 is the initial size of population at time $t = 0$

  r is the rate of growth and $r > 0$.

  t is time.

- Doubling time is the time that elapses for the population to double, denoted by d. If doubling time is given, find the rate of growth using this formula: gives $r = \dfrac{\ln 2}{d}$.

  Fun Fact: What is $n(d)$?

Exponential Decay

- The **half-life of a radioactive substance** is the time required for the amount of the substance to be reduced to half of its original amount.

- If m_0 is the initial mass of the radioactive substance with half-life h, then the mass remaining at time t is modeled by the function $\boxed{m(t) = m_0 e^{-rt}}$ where $r = \dfrac{\ln 2}{h}$ is a positive number.

- Other ways to model this with $\boxed{m(t) = m_0 e^{rt}}$ where $r < 0$.

  Fun Fact: What is $m(h)$?

General Guideline for Modeling with Exponential Functions

- Diagnostic: Find if exponential growth or decay.

- Use the values given at time zero to find the initial value.

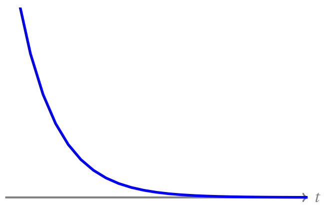
- Use the given output values at other input values to solve for r.

- Form the model. Use it to find the other values asked in the problem.

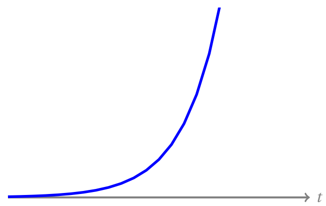
- Often in decay models (and sometimes in growth models, we are asked to find the time when the amount remaining (the growth) is given. Set the model's function $\underbrace{\text{equal}}_{=}$ to the given amount and solve an **exponential equation** to find **time**.

The End Behavior or Long-Term Behavior

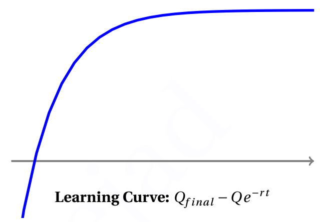
- In real life modeling, we typically see **three** options for graphs. (Figures are shown below.)
- The behaviour may be **asymptotic**, **exponential decay** or a **simple learning curve**; the behavior may be an **exponential growth**.



Exponential decay: $Q_0 e^{-rt}$

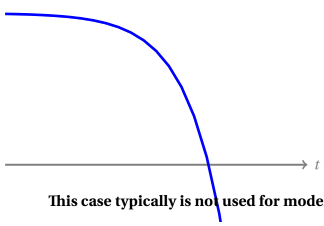


Exponential growth: $Q_0 e^{rt}$



Learning Curve: $Q_{final} - Qe^{-rt}$

- The following case is typically **NOT** used for modeling.



This case typically is not used for modeling a long term behavior: $Q_{final} - Qe^{rt}$

A Few Worked-out Examples

Example 1: A generous donor wishes to establish a fund to provide University of Kansas with an annual grant of \$50,000 beginning next year. If the fund will earn an interest rate of 9%/year compounded continuously, find the amount of the endowment the donor is required to make now. (Hint: The annual interest earned from the principal has to be equal to the annual grant amount.)

Solution:

$A(t) = Pe^{rt} \implies$ Interest$= Pe^{rt} - P$.

$t = 1$ year, P is the principal which is the variable we are solving for, $r = 0.09$ so:

$50,000 = Pe^{0.09} - P \implies P = \dfrac{50,000}{e^{0.09} - 1} \approx \boxed{530,931}$ dollars.

Example 2: **Archaeology and Anthropology:** The half-life of carbon-14 is 5,730 years.

(a) Explain the science of carbon dating in 2-4 sentences.

(b) Express the amount of carbon-14 remaining as a function of time, t years.

(c) How old is an object if the amount of carbon is reduced to 20% of its original amount?

Solution

(a) After an animal or a tree dies, they stop taking in Carbon 14 from atmosphere or their food resources. The carbon 14 starts decaying over time and we can model the decay by an exponential function. To find the time past the death, we use the measured amount of Carbon 14 and set up an equation to solve for time.

(b)

$Q(t) = Q_0 e^{rt}$ (The general continuous growth formula.)

$0.5Q_0 = Q_0 e^{r \cdot 5730}$ (Substitute the half-life for t and $0.5Q_0$ for $Q(t)$.)

$0.5 = e^{5730r}$ (Divide by Q_0 to isolate.)

$\ln(0.5) = 5730r$ (Take the natural log of both sides.)

$r = \dfrac{\ln(0.5)}{5730}$ (Solve for r.)

$\boxed{Q = Q_0 e^{\left(\frac{ln(0.5)}{5730}\right)t}}$ (Replace r in the formula.)

Note: $r = \dfrac{ln(0.5)}{5730}$ is a negative value.

(c)

Set up the equation: $0.2Q_0 = Q_0 e^{\left(\frac{ln(0.5)}{5730}\right)t} \underset{\text{Divide by} Q_0 \text{ and take log of both sides}}{\Longrightarrow}$

$\boxed{t = \dfrac{5730\ln(0.2)}{\ln(0.5)}}$

1. The number of widgets an employee can produce t months after starting to work at the widget factory is $Q(t) = 76 - 16e^{-0.3t}$ per day.

   (A) How many widget does the employee produce on their first day?

   (B) When an employee is **very experienced**, how many widgets per day can she be expected to produce?

   (C) What is the **average rate of change** in number of widget produced over the time interval $[0, 20]$.

2. Strontium 90 (Sr-90), a radioactive isotope of strontium, is present in the fallout resulting from nuclear explosions. It is especially hazardous to animal life, including humans, because, upon ingestion of contaminated food, it is absorbed into the bone structure. Its half-life is 27 years.

   (A) Assume the initial amount of Sr-90 is A_0. Quickly, compute the amount when time elapsed is given in the below table.

Time Elapsed	0	$\underbrace{27}_{\text{one half time}}$	$\underbrace{2\times 27}_{\text{two half times}} = 54$	$\underbrace{3\times 27}_{\text{three half times}} = 81$	$\underbrace{4\times 27}_{\text{four half times}} = 108$
Amount Remaining	A_0				

   (B) If the amount of Sr-90 in a certain area is found to be 5 times the "safe" level, find how much time must elapse before the safe level is reached.

3. How long does it take for an investment of \$2500 to become 4-fold its value if the interest rate is 8.7% compounded continuously?

4. Magnesium-27 has a half-life of 9.45 minutes. Suppose we are given a 85 mg sample of magnesium-27. In this problem, if you choose to round any of your values, round to at least two decimal places.

   (A) Assuming the amount of magnesium-27 remaining follows an exponential decay model, find a function $m(t)$ modeling the mass remaining after t minutes.

   (B) How much of the sample remains after 15 minutes?

   (C) After how many minutes will 10 mg of the sample remain?

Mini Projects: Choose one of the following mini projects that fits your interest best; seek others in your class who are in your major or are interested in the same project; complete the project with them in class; fill out the survey about the project on Canvas afterward.

5. **Aerospace Engineering, Mini Project 1:** As the air density decreases in higher altitudes the performance of the aircraft decreases. In Aerospace Engineering, the air density at altitude h is best approximated by an exponential decay model.

$$\rho(h) = \rho_0 e^{-nh}$$

where n is a constant, $\rho_0 = 23.77 \times 10^{-4}$ slugs$/ft^3$ is the air density at sea level, and h is the altitude from sea level.

(a) Why does air density decrease as the altitude increases?

(b) The air density at altitude $20,000\,ft$ is 12.67×10^{-4} slugs$/ft^3$; find n using this data point.

(c) Rewrite the formula by entering values of n and ρ_0.

(d) Compute the air density for the following altitudes using the formula you found in Part (c).

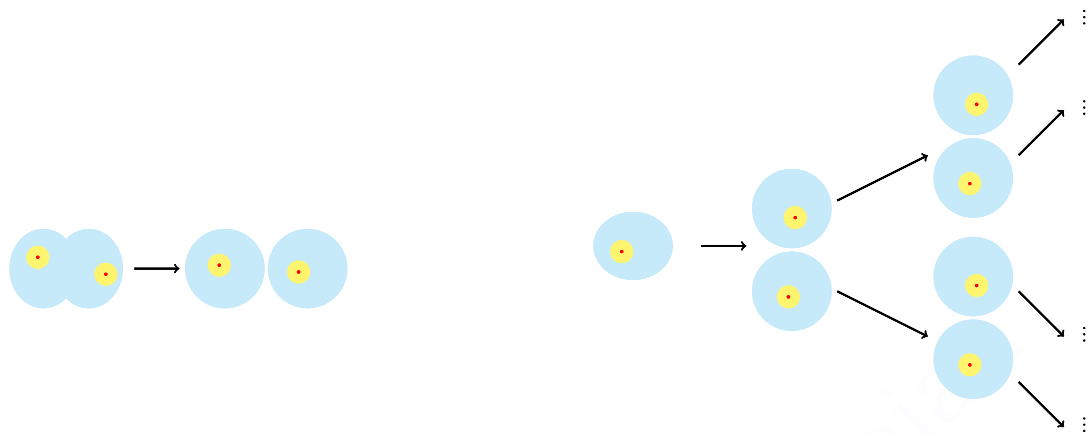
Altitude (ft)	Density (10^{-4} slugs$/ft^3$)
5,000	
10,000	
30,000	
40,000	

(e) Compare the results in Part (d) with the actual measurements in table below. What is your opinion on accuracy of the exponential decay model?

Altitude (ft)	Density (10^{-4} slugs$/ft^3$)
5,000	20.48
10,000	17.56
30,000	8.91
40,000	5.87

The creation of engineering mini projects is an ongoing collaboration with school of engineering. The Aerospace mini project in this document was created in collaboration with professor Emily Arnold (Aerospace Engineering).

6. **Biological Sciences, Mini Project 2:**
A culture of Bacteria starts with 8700 bacteria. After one hour the count is $12,000$.



(a) The growth of the culture after t hours is modeled by a exponential model $n(t) = n_0 e^{rt}$; what is the value of n_0?

(b) What is the rate of growth, r, of the population?

(c) Rewrite the model by replacing the numerical values in the formula; find the function that models the number of bacteria $n(t)$ after t hours.

(d) Find the number of bacteria after 4 hours.

(e) After how many hours, does the number of bacteria double?

(f) Let T_{Double} denote the time you found in Part (e); complete the following table.

Time elapsed (hr)	Number of Bacteria in the Dish
0	
T_{Double}	
$2 \times T_{Double}$	
$3 \times T_{Double}$	

(g) The growth of bacteria follows exponential model in the beginning but it eventually slows down. What conditions slow down the exponential growth of the culture?

7. **Chemistry, Archaeology or Forensic Sciences, Mini Project 3:** Carbon-14 has a half-life of 5,730 years. Carbon-14 decays exponentially; following model $A(t) = A_0 e^{-rt}$.

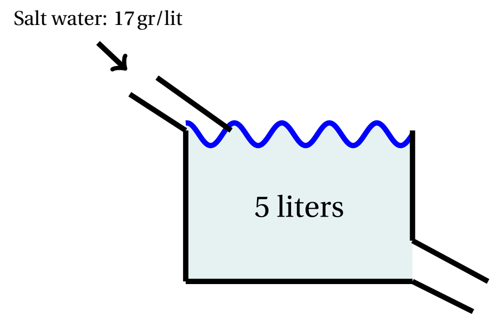
(a) What is the difference between Carbon-12 and Carbon-14? Do they both decay?

(b) Suppose that the initial amount of Carbon-14 found in an animal when it dies is A_0. Write an exponential model representing the **amount** of Carbon-14, $A(t)$, found in a fossil t years after the animal dies.

(c) What is the percentage of the **amount** remaining after 900 years has passed?

(d) When it is found, a fossil has 17% of the amount of carbon-14 the animal had when it died. How old is the fossil? (Round your answer to the nearest year.)

8. **Chemical Engineering, Mini Project 4:** A tank is filled with 5 liters of water. Salt water of concentration 17 grams per liter is constantly pumped into the tank and a well-mixed solution exits the tank at the same rate. The quantity of salt in the tank as a function of time is modeled by



$$Q(t) = Q_{\text{final}} - 70e^{-0.04t}$$

where $Q(t)$ is the amount of salt, in grams, when t minutes have passed.

(A) What is the concentration of salt in the **tank** water in the long run?

(B) What is the amount of salt in the tank in the long run?

(C) What is the horizontal asymptote of the function $Q(t) = Q_{\text{final}} - 70e^{-0.04t}$?

(D) Find the value of Q_{final} and rewrite the function using that value.

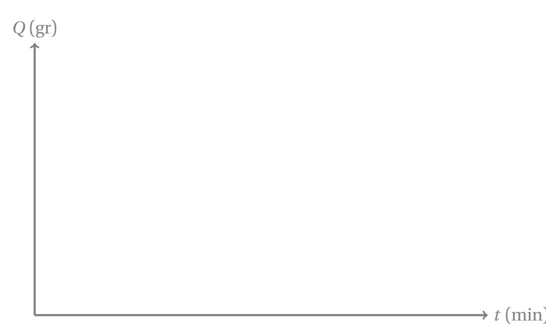
(E) What is the initial amount of salt in the tank?

(F) What is the amount of salt in the tank at $t = 20$ minutes?

(G) When is the amount of salt in the tank equal to 76 grams?

(H) Using the answers in Parts B, D, E, F and G, complete the table and plot the graph. Does the data make sense?

Time (min)	$Q(t)(gr)$
0	
20	
	76
1000	

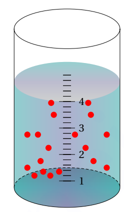


Pre-project notes: A chemical reaction is the process of changing some compounds (reactants) into some other compounds (product).

$$\text{For Example, } \underbrace{AB}_{\text{Reactant}} \to \underbrace{A+B}_{\text{Products}} \text{ or } \underbrace{A+B}_{\text{Reactants}} \to \underbrace{AB}_{\text{Product}} \text{ or } \cdots$$

In a chemical reaction, a more formal definition of reactant(s) is the compound(s) whose concentration is decreasing and the product is the compound whose concentration is increasing.

9. **Chemistry, Mini Project 5:**



In a chemical reaction, the **Integrated Rate Law** is an expression for the concentration, measured in units of molar (M), of a reactant or a product in terms of time; the concentration of a compound in a mixture when measured in molar is also called **molarity**. Let

$[A]$: The concentration function of the chemical reactant A
$[A]_t$: The concentration or molarity of the chemical reactant A at time t
$[A]_0$: The concentration or molarity of the chemical reactant reactant A at time 0
k: The rate constant is unique for a given chemical reaction at a certain temperature
$\frac{\Delta[A]}{\Delta t}$: The average rate of change in concentration of A

(A) The notation $[A]_t$ and $[A]_0$ are different in mathematics. How do you suggest to denote $[A]_t$ and $[A]_0$ mathematically?

(B) The rate of a reaction is related to the concentration of the reactant in the reaction. Depending on the chemical reaction the rate can follow one of the following

Order of Reaction: 0	Order of Reaction: 1	Order of Reaction: 2
$\frac{\Delta[A]}{\Delta t} = -k$	$\frac{\Delta[A]}{\Delta t} = -k[A]$	$\frac{\Delta[A]}{\Delta t} = -k[A]^2$

In future math courses, you will learn that the above relation can be solved in term of an expression of $[A]$ and t as follows. For now, **in the blank row, explain how the two solutions are equivalent using Algebra**.

Order of Reaction	0^{th}	1^{st}	2^{nd}
	$[A]_t - [A]_0 = -kt$	$\ln\left(\frac{[A]_t}{[A]_0}\right) = -kt$	$\frac{1}{[A]_t} - \frac{1}{[A]_0} = kt$
Explain why each pair are equivalent in each column:			
	$[A]_t = -kt + [A]_0$	$\ln([A]_t) = -kt + \ln([A]_0)$	$\frac{1}{[A]_t} = kt + \frac{1}{[A]_0}$

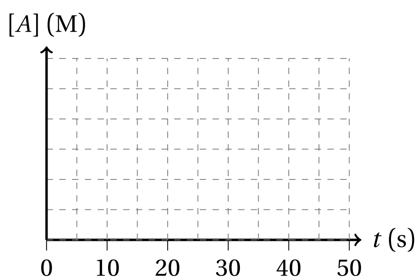
To find the order of a reaction, we plot $[X]_t$ in Part (C), $\ln([X]_t)$ and $\dfrac{1}{[X]_t}$ versus time to determine which is linear.

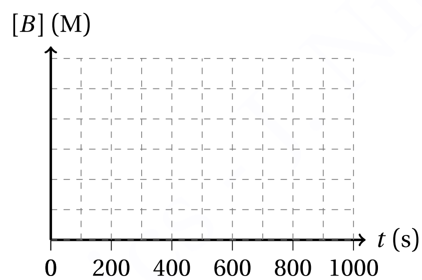
(C) ① Plot each of the following reactant molarity versus time data. ② Determine which plot is linear. ③ Which reaction is of order 0? ④ What is the slope of the line in that case? ⑤ What is the rate constant, k, for that reaction? ⑥ How long does it take for the reaction of order 0 to consume half of the chemical reactant? ⑦ After half of the chemical reactant is consumed, how long does it take for the remaining chemical reactant to be halved again?

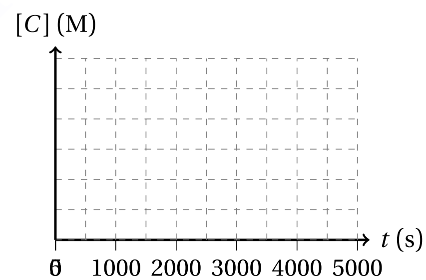
Time (s)	[A] (M)
0	0.20
10.0	0.17
20.0	0.14
30.0	0.11
40.0	0.08
50.0	0.05

Time (s)	[B] (M)
0	15
200	9.6
400	6.2
600	4
800	2.5
1000	1.6

Time (s)	[C] (M)
0	1
1000	0.11
2000	0.061
3000	0.041
4000	0.031







[1]

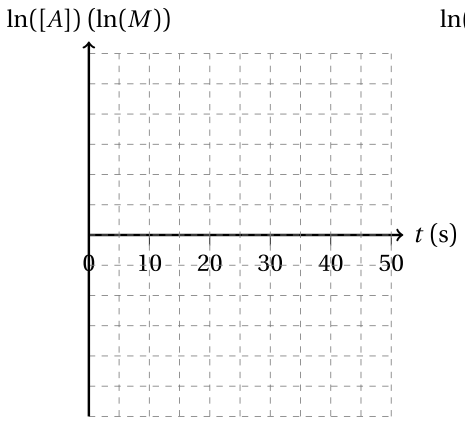
(D) ① Use the data from Part (C), when needed, and complete each of the following tables. ② Plot each of the following logarithmic reactant molarity versus time data. ③ Determine which plot is linear. ④ Which reaction is of order 1? ⑤ What is the slope of the line in that case? ⑥ What is the rate constant, k, for that reaction? ⑦ What is half-life of the chemical reactant in that case?

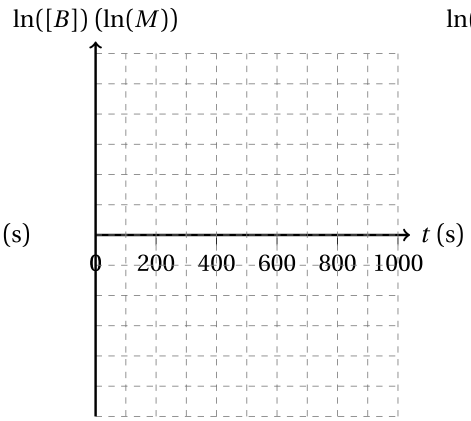
Time (s)	$\ln([A])$ $(\ln(M))$
0	
10.0	$\ln(0.17) = -1.772$
20.0	$\ln(0.14) = -1.966$
30.0	$\ln(0.11) = -2.207$
40.0	$\ln(0.08) = -2.526$
50.0	$\ln(0.05) = -2.996$

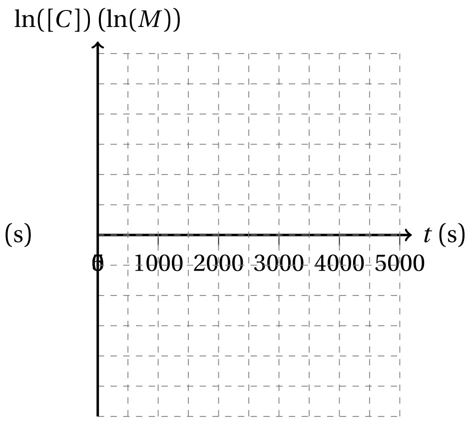
Time (s)	$\ln([B])$ $(\ln(M))$
0	$\ln(15) = 2.708$
200	$\ln(9.6) = 2.262$
400	$\ln(6.2) = 1.825$
600	$\ln(4) = 1.386$
800	$\ln(2.5) = 0.916$
1000	

Time (s)	$\ln([C])$ $(\ln(M))$
0	$\ln(1) = 0$
1000	
2000	$\ln(0.061) = -2.797$
3000	$\ln(0.041) = -3.194$
4000	$\ln(0.031) = -3.474$

[1] Note the best practice to find the line for each linear table in Parts (C), (D) and (E), is to find the line of best fit for this data using your TI 84 or an excel sheet. That is what you will need to do in a Chemistry lab when empirical data is obtained and analyzed. To shorten the mini-project, we are using synthesized data and asking you to pick any two data points and assume that the line of best fit is going through them. In your Chem lab, you will be given instructions on how find accurate equation of line.





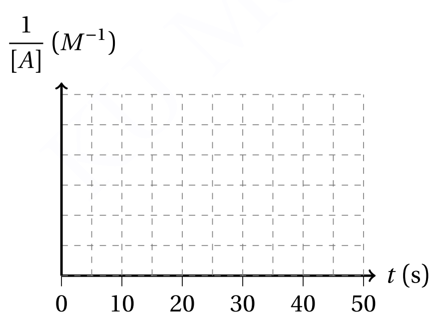


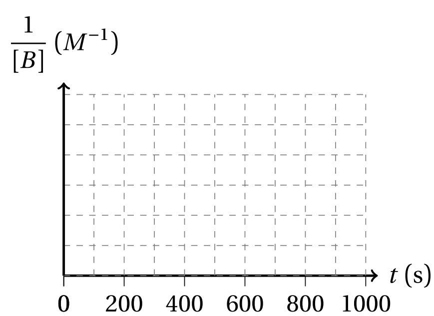
(E) Use the data from Part (C), when needed, and complete each of the following tables. Plot each of the following reciprocal reactant molarity versus time data. Determine which plot is linear. Which reaction is of order 2? What is the slope of the line in that case? What is rate constant, k, for that reaction?

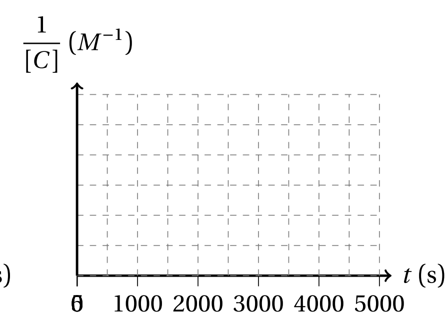
Time (s)	$\frac{1}{[A]}$ (M^{-1})
0	
10.0	$\frac{1}{0.17} = 5.882$
20.0	$\frac{1}{0.14} = 7.143$
30.0	$\frac{1}{0.11} = 9.091$
40.0	
50.0	

Time (s)	$\frac{1}{[B]}$ (M^{-1})
0	
200	$\frac{1}{9.6} = 0.104$
400	$\frac{1}{6.2} = 0.161$
600	$\frac{1}{4} = 0.25$
800	$\frac{1}{2.5} = 0.4$
1000	$\frac{1}{1.6} = 0.625$

Time (s)	$\frac{1}{[B]}$ (M^{-1})
0	
0	$\frac{1}{1} = 1$
2000	$\frac{1}{0.061} = 16.393$
3000	$\frac{1}{0.041} = 24.39$
4000	$\frac{1}{0.031} = 32.258$







The creation of STEM mini projects is an an ongoing collaboration with Chemistry and Biology Departments. The Chemistry mini project in this document was created in collaboration with Professor Erin Burger-Dunn.